

Rare fluctuations and the high-energy limit of the S -matrix in QCD

Edmond Iancu

*Service de Physique Théorique, CEA Saclay
91191 Gif-sur-Yvette cedex, France*

and

A.H. Mueller¹

*Department of Physics, Columbia University
New York, New York 10027, USA*

Abstract

We argue that one cannot correctly calculate the elastic scattering S -matrix for high-energy dipole-dipole scattering, in the region where S is small, without taking fluctuations into account. The relevant fluctuations are rare and unimportant for general properties of inelastic collisions. We find that the Kovchegov equation, while giving the form of the S -matrix correctly, gives the exponential factor twice as large as the result which emerges when fluctuations are taken into account.

¹This research is supported in part by the US Department of Energy.

1 Introduction

The topic of this paper is understanding the role that fluctuations play when unitarity corrections become important in high-energy scattering. We focus our attention on dipole-dipole scattering because the issues are a bit sharper there. We can perhaps motivate our discussion by the following observations. At very high energy the typical (mean) configuration of a dipole's light-cone wavefunction is a Color Glass Condensate [1–4], a state characterized by a saturation [5, 6] momentum, Q_s , and having high occupancy for all gluonic levels of momentum less than or equal to Q_s , which depends on the rapidity, y , of the parent dipole. Suppose we scatter two such evolved dipoles at zero impact parameter, in the center of mass frame and at relative rapidity Y . Then, if we use just these typical configurations, of the condensate type, to compute the S -matrix for the scattering, the result will be proportional to $\exp\{-const Q_s^2(Y/2)r_0^2/\alpha_s^2\}$, where r_0 is the size of the parent dipoles serving as the seeds for Color Glass Condensates which collide. The $Q_s^2 r_0^2$ part of this formula is purely geometric; it is the number of, roughly, independent parts of one of the condensates when viewed on a scale $\Delta x_\perp \sim 1/Q_s$. A more complete discussion of condensate-condensate scattering is given in section 3.3 with the S -matrix given in eq. (25). On the other hand the Kovchegov equation [7] gives an S -matrix (see eq. (14)) which is much larger than (25), being proportional to $\exp\{-const \ln^2(Q_s^2 r_0^2)\}$. Why do eqs. (14) and (25) disagree and which of these answers is correct ?

The problem is in the use of typical, or mean, configurations to estimate the elastic scattering S -matrix. If one collides in the center of mass frame two Color Glass Condensates, then the exponent we arrived at, $Q_s^2 r_0^2/\alpha_s^2$, corresponds to the average number of gluons produced in the collision. The condensate description of the wavefunction is a good starting point for a typical (inelastic) collision, but it need not be correct for evaluating a very small observable like the elastic S -matrix whose evaluation can be (and is) dominated by much rarer configurations. As we shall see the parts of the wavefunction which dominate the S -matrix are very dependent on the frame in which we view the scattering. In the center of mass frame, and in the high-energy regime where S is very small, the S -matrix is dominated by wavefunction configurations which have relatively few gluons [8] and which are not in a condensate state. In an asymmetric frame, the parent dipole having the lower momentum evolves into a state with few gluons while the higher momentum dipole evolves into a condensate, but into a condensate

having a lower than normal saturation momentum.

What about the result (14) coming from the Kovchegov equation? While the functional form of the solution [3, 9] is fine, we find that the Kovchegov equation misses the correct evaluation of the exponent of S by a factor of two. Again the problem is that the Kovchegov equation does not treat fluctuations properly. The crucial ingredient in deriving the Kovchegov equation is the replacement of $S_Y^{(2)}(x_\perp - z_\perp, z_\perp - y_\perp)$, the scattering S -matrix for a two-dipole state (dipoles of sizes $x_\perp - z_\perp$ and $z_\perp - y_\perp$) by the product $S_Y(x_\perp - z_\perp)S_Y(z_\perp - y_\perp)$. Such a replacement is true only in the absence of fluctuations in the light-cone wavefunction of the target. Contrary to a rather widely held belief this replacement is not justified by large N_c behavior since, as we shall see below, the relevant fluctuations concern a suppression of gluons in certain momentum regions of the wavefunction, an issue which is independent of N_c . Despite these problems, the Kovchegov equation is probably the best one can do with respect to a simple equation which naturally imposes unitarity on BFKL evolution.

Much of what we say here has been anticipated in some early works [8, 10] on the study of unitarity in dipole-dipole scattering. The issue of fluctuations was reasonably well understood although at that time it was not possible to achieve the level of analytical precision that we are able to present here. The detailed numerical studies done by Salam [10] correspond to a (numerical) evaluation of the Balitsky equation [11] or, equivalently, of the functional evolution equation for the Color Glass Condensate [1, 4, 12, 13], rather than an evaluation of the Kovchegov equation.

Finally, it must be admitted that we are not able to prove that the wavefunction configurations which we suggest are dominant are in fact so. Certainly, the configurations that we shall find are better than those chosen by the Kovchegov equation since our S -matrix is much bigger. And we do give arguments in sections 3.6 and 4 to the end that natural modifications of our choice of configurations lead to a smaller S -matrix. Nevertheless, it is difficult to classify all the possible configurations so as to be sure that we have made the right choice. In this context it should be noted that the coefficient of the $(\bar{\alpha}_s Y)^2$ term in the exponent of our result (31) is about a factor of 1.7 higher than the numerical evaluation in Ref. [8]. We do not understand the resolution of the discrepancy.

In Sec. 2 we give a rapid derivation of the Kovchegov equation, and of its prediction for the high-energy behavior of the S -matrix for dipole-dipole scattering in the regime where S is very small. The usual Levin–Tuchin result

[3, 9] is recovered.

In Sec. 3, we investigate dipole-dipole scattering at very high energies in the center of mass (CM) frame and at fixed impact parameter. In Sec. 3.1 we review the BFKL-dipole picture. In Sec. 3.2, we show that using BFKL evolution for the dipole wavefunctions, and taking into account only average properties like their respective dipole number densities, leads to an S -matrix which is too small, even though we work in a rapidity region where BFKL evolution is correct for average properties of the wavefunction. In Sec. 3.3 we collide two Color Glass Condensates, and again find an S -matrix which is unrealistically small. In Sec. 3.4 we describe which (rare) configurations lead to the result of the Kovchegov equation in a natural asymmetric frame. In Sec. 3.5 we describe a set of rare fluctuations which give a much larger S -matrix and which we suggest dominate very high-energy scattering in the CM frame. In Sec. 3.6 we give (incomplete) arguments as to why the configurations we have chosen are optimal.

In Sec. 4, we transform our center of mass picture to an arbitrary frame and describe the picture which emerges there.

2 The Kovchegov equation

The Kovchegov equation [7] is probably the best “simple” equation for dealing with the onset of unitarity in high energy, but weak coupling, scattering in QCD. The argument for the Kovchegov equation is easily stated. Consider the high-energy scattering of a dipole, consisting of a quark at x_\perp and an antiquark at y_\perp , on a target which may be another dipole or a more complex hadron or nucleus. It is convenient to view the scattering in a frame where the dipole is going along the negative z -axis (left-moving) and the target is going along the positive z -axis (right-moving), and where almost all of the rapidity, Y , of the scattering is taken up by the right-moving system. (This is sometimes referred to as the “dipole frame”.) In this frame the process takes the form of an elementary (unevolved) dipole of size $x_\perp - y_\perp$ scattering in the field of a (highly evolved) target. If $S_Y(x_\perp, y_\perp)$ is the S -matrix for the scattering we define the amplitude \mathcal{N}_{xy} by

$$\mathcal{N}_{xy} = 1 - S_Y(x_\perp, y_\perp). \quad (1)$$

Now suppose we increase Y by a small amount. If this increase is given to the target, then in order to calculate the change in S_Y , it is necessary to

calculate how the wavefunction of the target changes with Y . This change can be calculated using a functional equation derived by Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) [1, 4, 12, 13]. However, in order to derive the Kovchegov equation it is simpler to keep the rapidity of the target fixed and put the small change of rapidity into the left-moving elementary dipole. The dipole now has a small probability of emitting a gluon due to this change of rapidity. If the gluon is in the wavefunction of the dipole at the time it scatters on the target, then what scatters is a quark-antiquark-gluon system which, in the large N_c limit, can be viewed as a system of two dipoles. (One of these dipoles is the original quark and the antiquark part of the gluon while the other dipole is the quark part of the gluon and the original antiquark.) If the gluon is not in the wavefunction at the time of the scattering, it can be viewed as the “virtual” term which decreases the probability that the quark-antiquark pair remain a simple dipole, compensating the probability for the two-dipole state. One can put this in formulae as

$$\begin{aligned} \frac{\partial}{\partial y} S_Y(x_\perp, y_\perp) &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\times \left[S_Y^{(2)}(x_\perp, z_\perp, y_\perp) - S_Y(x_\perp, y_\perp) \right], \end{aligned} \quad (2)$$

where $\bar{\alpha}_s = \alpha_s N_c / \pi$, z_\perp is the transverse coordinate of the emitted gluon, and $S_Y^{(2)}$ stands for the scattering of the two dipole left-moving system on the target.

If the target is homogeneous on a scale large compared to $x_\perp - y_\perp$, then it is convenient to view S_Y as a function of $x_\perp - y_\perp$ and $b_\perp = \frac{x_\perp + y_\perp}{2}$, and $S_Y^{(2)}$ as a function of $x_\perp - y_\perp$, $z_\perp - y_\perp$ and b_\perp . Suppressing the b_\perp -dependence, eq. (2) becomes

$$\begin{aligned} \frac{\partial}{\partial y} S_Y(x_\perp - y_\perp) &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\times \left[S_Y^{(2)}(x_\perp - z_\perp, z_\perp - y_\perp) - S_Y(x_\perp - y_\perp) \right]. \end{aligned} \quad (3)$$

This is not yet the Kovchegov equation, rather it is part of a set of equations derived by Balitsky [11], and which also follows from the functional evolution equation in Refs. [1, 2, 13]. Eq. (3) is very difficult to use because a solution for the Y -dependence of S_Y requires knowing $S_Y^{(2)}$. One obtains the

Kovchegov equation if one further assumes

$$S_Y^{(2)}(x_\perp - z_\perp, z_\perp - y_\perp) = S_Y(x_\perp - z_\perp)S_Y(z_\perp - y_\perp), \quad (4)$$

which is a sort of mean field approximation for the gluonic fields in the target. In his original papers [7], Kovchegov has obtained eq. (4) for the case that the target is a large nucleus; in that case, this equation should be a good approximation so long as Y is not so large as to cause S to be dominated by nuclear density fluctuations. However for general targets one cannot expect (4) to be exact, as we shall see in the next section. Nevertheless, (4) may be a reasonable approximation as unitarity corrections are just becoming important, and in any case it leads to an interesting equation.

Specifically, using the approximation (4) in eq. (3) gives

$$\begin{aligned} \frac{\partial}{\partial y} S_Y(x_\perp - y_\perp) &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\quad \times [S_Y(x_\perp - z_\perp)S_Y(z_\perp - y_\perp) - S_Y(x_\perp - y_\perp)], \end{aligned} \quad (5)$$

or, equivalently,

$$\begin{aligned} \frac{\partial}{\partial y} \mathcal{N}_{xy} &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\quad \times [\mathcal{N}_{xz} + \mathcal{N}_{zy} - \mathcal{N}_{xy} - \mathcal{N}_{xz} \mathcal{N}_{zy}], \end{aligned} \quad (6)$$

which are two common forms of the Kovchegov equation [7]. Eq. (6) is the more useful equation when scattering is weak. In that case the quadratic term in (6) may be dropped and the dipole version of the BFKL equation results. Eq. (5) is easier to use when S is small, and unitarity corrections have become very important.

But in the general case, when neither S nor \mathcal{N} is small, eqs. (5) or (6) are difficult to deal with analytically. There are a number of good numerical evaluations [14–19] of the Kovchegov equation which do cover the region where S is neither close to zero nor close to one. Here, however, we are interested only in the high-energy regime where unitarity corrections are very important, so S is small indeed, and the solution to eq. (5) can be evaluated analytically. If $S_Y(x_\perp - y_\perp)$ is very small, then the quadratic term in S on the right-hand side of (5) is much smaller than the linear term unless either $(x_\perp - z_\perp)^2$ or $(z_\perp - y_\perp)^2$ is as small as $Q_s^{-2}(Y)$. We have introduced the saturation momentum, $Q_s(Y)$, which is an intrinsic scale of

the target (characteristic of the gluon density there), and which marks the scale at which a dipole scattering off the target makes the transition from weak ($r_\perp \ll 1/Q_s$) to strong ($r_\perp \gg 1/Q_s$) interactions (r_\perp is the size of the dipole). More precisely, it is common to define $Q_s(Y)$ by the equation $S_Y(r_\perp) = 1/e$ when $r_\perp = 2/Q_s$.

The regime of interest here corresponds to $r_0 \equiv |x_\perp - y_\perp| \gg 2/Q_s$. Then, the right-hand side of (5) is dominated by the logarithmic regions of integration where one has either

$$4/Q_s^2 \ll (x_\perp - z_\perp)^2 \ll r_0^2, \quad (7)$$

or

$$4/Q_s^2 \ll (z_\perp - y_\perp)^2 \ll r_0^2. \quad (8)$$

One easily finds

$$\frac{\partial}{\partial y} \ln S_Y(r_0) \simeq -\bar{\alpha}_s \int_{Q_s^{-2}(Y)}^{r_0^2} \frac{dr^2}{r^2} = -\bar{\alpha}_s \ln[Q_s^2(Y)r_0^2]. \quad (9)$$

Now, we know that (see, e.g., [4])

$$\ln[Q_s^2(Y)r_0^2] = c\bar{\alpha}_s(Y - Y_0) + \dots \quad (10)$$

where Y_0 is such that $Q_s(Y_0) \sim 1/r_0$, and [5, 20]

$$c \equiv \frac{2\chi(\lambda_0)}{1 - \lambda_0} \simeq 4.883. \quad (11)$$

In the equation above, λ_0 is the solution to

$$\chi(\lambda_0) = -\chi'(\lambda_0)(1 - \lambda_0), \quad (12)$$

where

$$\chi(\lambda) = \psi(1) - \frac{1}{2}\psi(\lambda) - \frac{1}{2}\psi(1 - \lambda) \quad (13)$$

is the usual BFKL [21, 22] eigenvalue function. The omitted terms on the right-hand side of (10) have only a logarithmic dependence on Y . Using (10) in (9) one finds [3, 4, 9]

$$S_Y(r_0) = e^{-\frac{c}{2}\bar{\alpha}_s^2(Y - Y_0)^2} S_{Y_0}(r_0), \quad (14)$$

where $S_{Y_0}(r_0) \sim 1$.

Because we have not kept $\ln(Y - Y_0)$ terms on the right-hand side of (10), there are some missing linear terms in Y in the exponent of eq. (14). Our focus in this paper will be primarily on the quadratic term, and its coefficient, in the exponent of (14).

Eq. (14) gives the standard result in the literature. We have gone through such a detailed “derivation” of (14) because the main point of the present paper is to argue that (14) is in fact not quite right. The problem, as we shall see, is not with the derivation of (14) from (5), but with the mean field approximation contained in (4). Deep in the saturation region, where S is very small, fluctuations away from average configurations are actually dominant for the calculation of S .

3 Dipole-dipole scattering in the CM system

In this section we consider dipole-dipole scattering in the center of mass frame. Of course, the scattering cross section cannot depend on the frame in which it is calculated, however, the form that the calculation takes can be very different in different frames in a non-covariant gauge calculation. In the first three parts of this section we proceed in a qualitative manner, in order that the physics issues not be obscured by technical details. In the final three parts of this section we proceed more technically and arrive at an evaluation of the S -matrix which disagrees with (14), being a factor of two smaller in the exponent.

3.1 The BFKL approximation

Suppose we scatter two dipoles, each of size r_0 , at a relative rapidity Y . Then in a frame where one of the dipoles has rapidity y and the other has rapidity $Y - y$ the BFKL approximation gives [23, 24]

$$\sigma(Y) = \int \frac{d^2r_1 d^2r_2}{4\pi^2 r_1^2 r_2^2} n(r_0, r_1, y) n(r_0, r_2, Y - y) \sigma^{(0)}(r_1, r_2) \quad (15)$$

where $n(r_0, r_1, y)$ is the number density of radiated dipoles of size r_1 in the wavefunction of a parent dipole of size r_0 , within a rapidity interval equal to y . Furthermore, $\sigma^{(0)}$ is the lowest order dipole-dipole scattering cross

section, given by

$$\sigma^{(0)}(r_1, r_2) = 2\pi\alpha_s^2 r_<^2 \left(1 + \ln \frac{r_>}{r_<}\right), \quad (16)$$

where $r_<$ is the smaller of r_1, r_2 and $r_>$ is the larger of r_1, r_2 . In the saddle point approximation to the BFKL solution,

$$n(r_0, r_1, y) \approx \frac{r_0}{2r_1} \frac{e^{(\alpha_P-1)y}}{\sqrt{\frac{7}{2}\alpha_s N_c \zeta(3)y}} \exp\left\{-\frac{\ln^2(r_0/r_1)}{14\bar{\alpha}_s \zeta(3)y}\right\} \quad (17)$$

leading to

$$\sigma(Y) \approx 4\pi\alpha_s^2 r_0^2 \frac{e^{(\alpha_P-1)Y}}{\sqrt{\frac{7}{2}\alpha_s N_c \zeta(3)Y}}. \quad (18)$$

In these equations, $\alpha_P - 1 = 4\bar{\alpha}_s \ln 2$ is the BFKL ‘intercept’. What we are interested in here is not so much the precise formulae contained in eqs. (15)–(18), but rather the qualitative picture. To that end we rewrite the cross-section, eq. (18), as

$$\sigma(Y) = \pi r_0^2 \alpha_s^2 n^2(r_0, r_0, Y/2) C(\alpha_s Y) \quad (19)$$

where

$$C(\alpha_s Y) = 8\sqrt{\frac{7}{2}\alpha_s N_c \zeta(3)Y} \quad (20)$$

is an uninteresting and slowly varying prefactor. Eq. (19) gives the cross section in terms of an elementary dipole–dipole cross section, $\alpha_s^2 \pi r_0^2$, times the number of evolved dipoles in the wavefunctions of each of the colliding parent dipoles. Eq. (19) is a reliable formula so long as σ is well below πr_0^2 . When σ approaches πr_0^2 , unitarity corrections become important and it is to this topic that we next turn.

3.2 The onset of unitarity corrections

Let Y_0 be the rapidity where $\sigma(Y_0)$ is equal to πr_0^2 . Then, roughly,

$$Y_0 \simeq \frac{1}{\alpha_P - 1} \ln \frac{1}{\alpha_s^2}, \quad (21)$$

where the neglected terms in (21) are of size $\frac{1}{\alpha_P-1} \ln \ln 1/\alpha_s$. Eq. (17) gives the average number of evolved dipoles in the parent dipole wavefunction so long as $y < Y_0$, at which point saturation effects in the dipole wavefunction become important. (The fact that the wavefunction of a dipole does not receive significant saturation corrections in the region $Y \ll Y_0$ [25] follows from eq. (15), which if we take y small requires that the result (18) come completely from $n(r_0, r_2, Y)$. This also shows that, when the scattering is seen in an *asymmetric* frame, unitarity corrections and saturation effects in the wavefunction of the evolved dipole start to be important at the same rapidity², namely at $Y = Y_0$.)

Now consider dipole–dipole scattering in the CM frame with the two parent dipoles scattering at zero impact parameter and at rapidity $Y \lesssim 2Y_0$. What is the S -matrix for such a collision? Of course when $Y > Y_0$ we expect very strong unitarity corrections to the scattering. However, it should be reasonable, so long as $Y < 2Y_0$, to assume that the individual wavefunctions of the colliding dipoles are still given by BFKL evolution (since $y = Y - y = Y/2 < Y_0$). Now S_Y^2 has the interpretation of being the probability that no interaction take place in the collision. It is tempting to estimate S_Y^2 by requiring that no evolved dipole in one wavefunction interact with any dipole in the other wavefunction. This gives

$$\ln S_Y^2(r_0) \simeq -c_0 \alpha_s^2 n^2(r_0, r_0, Y/2) \quad (22)$$

where the constant factor c_0 is not under control. When $Y \simeq 2Y_0$, using (17) and (21) in (22) gives

$$S_{2Y_0}(r_0) \simeq e^{-c/\alpha_s^2} \quad (23)$$

which is much smaller than what one would obtain using the result (14) from the Kovchegov equation along with the estimate (21).

Eq. (23) is clearly not right. The reason for such a failure is the fact that the above calculation of the S -matrix in the presence of unitarity corrections, cf. eqs. (22)–(23), has included only *typical* configurations in the wavefunctions of the parent dipoles. Of course, the use of typical configurations is correct as long as the energy is not too high ($Y < Y_0$), so that one can restrict oneself to the single scattering (or “single pomeron exchange”)

²This is in agreement with the discussion in Sec. 2, where we have seen that, in the dipole frame, the critical rapidity for strong scattering Y_0 is related to the saturation scale: $Q_s(Y_0) \sim 1/r_0$, cf. eq. (10).

approximation, eq. (18). But at higher energies ($Y > Y_0$), where multiple collisions *are* important, the wavefunction correlations among the dipoles which scatter simultaneously play a crucial role, and lead to a result for S which is very different from eqs. (22)–(23). So long as $Y < 2Y_0$, and for center of mass scattering, the relevant correlations are correctly described by BFKL evolution, but this has to be applied to the detailed dynamics of all the dipole configurations, and not only to average properties, like the mean dipole number density (17).

The correct calculation which sums up “multiple pomeron exchanges” and replaces eqs. (22)–(23) in the regime where $Y_0 < Y < 2Y_0$ is actually known. This has been originally developed within the color dipole picture [23, 24, 25], and recently rederived within the color glass formalism [26]. This is a rather complex calculation which cannot be analytically completed, but has been implemented numerically by Salam [10]. As mentioned in the Introduction, the numerical results thus obtained [8] are consistent with the functional form in eq. (14), but with a smaller coefficient in the exponent. As we shall see later in this section, such a smaller coefficient emerges indeed when some rare configurations in the wavefunction are taken into account.

3.3 The collision of two Color Glass Condensates

In order to emphasize that typical wavefunction configurations can give very wrong answers when S is small, we now estimate, from typical configurations, the S -matrix when $Y \gg Y_0$. Then, in the CM frame, what collide are two Color Glass Condensates, each characterized by a saturation momentum $Q_s(Y/2)$. The number of gluons having $k_\perp \sim Q_s$ in each wavefunction is

$$N \simeq c_1 \frac{1}{\alpha_s^2} Q_s^2(Y/2) r_0^2 \quad (24)$$

where, as before, r_0 is the size of the parent dipole in each of the colliding condensates. c_1 is a constant which is not important for our purposes. Again interpreting S_Y^2 as the probability that no collision takes place one easily gets

$$S_Y \simeq \exp \left\{ -c_2 \frac{1}{\alpha_s^2} Q_s^2(Y/2) r_0^2 \right\} \quad (25)$$

where the factor $Q_s^2 r_0^2$ in (24) and (25) counts the number of independent regions available for gluons, with $k_\perp \sim 1/Q_s$, to occupy in an area πr_0^2 .

Alternatively, the result in eq. (25) could be obtained by using the formalism developed in Ref. [26] for the scattering between two color glasses, and focusing on the typical configurations, of the condensate type, in both dipoles.

When $Y \rightarrow 2Y_0$, $Q_s^2(Y/2) \rightarrow Q_s^2(Y_0) \simeq 1/r_0^2$, and eq. (25) agrees with the previous estimate in eq. (23). Eq. (25) of course must also be wrong because it gives an S -matrix which is much too small. The problem with (25) is the same as with (23): rare, rather than typical, configurations of the wavefunction dominate the evaluation of the S -matrix in center of mass scattering when $Y > Y_0$.

3.4 Rare vs typical fluctuations in the Kovchegov equation

In order to better appreciate the fact that the role played in the scattering by the various parts of the wavefunction depends on the choice of frame we note that, even if we restrict ourselves to typical configurations, as in Secs. 3.2 and 3.3, the final result for the S -matrix can be very different if the calculation is performed in a different frame. For instance, let us choose the asymmetric frame previously used in Sec. 2, in which one of the participants in the collision is an elementary dipole while the other participant carries most of the total rapidity, and thus is highly evolved. If in this frame we compute S_Y as the scattering between the elementary dipole and the typical configuration in the energetic projectile, which is a condensate, one obtains [3] the same result as from the Kovchegov equation, namely Eq. (14). (Indeed, one can view Eqs. (7) and (8) as imposing the condensate condition on the wavefunction of the high energy projectile in our discussion in Sec. 2.) Although this result is not quite right, as we shall see in the next sections, it is nevertheless much larger (and thus closer to the correct result) than the result obtained by working in the center of mass frame, Eq. (25).

This discussion helps explain why the Kovchegov equation provides a qualitatively correct result although its derivation focuses on typical configurations, as we have seen in Sec. 2. But this also shows that if one tries to interpret the result of the Kovchegov equation in a different frame, like the center of mass frame, this interpretation will generally correspond to some rare configurations, although not necessarily the optimal ones.

Besides the “dipole frame” discussed above and in Sec. 2, where the relevant configurations are those of the condensate, there is another frame

in which the configurations pertinent to the Kovchegov equation are easily identified. For two parent dipoles of the same size r_0 , this is the frame in which the rapidity is equal to $Y - Y_0$ for the right-mover and to $-Y_0$ for the left-mover. We shall assume that the left-mover is subjected to normal evolution, as described by the JIMWLK equation, in the whole rapidity interval from $-Y_0$ to 0. Thus, at the time of scattering, this is a Color Glass Condensate with saturation scale $Q_s(Y_0) = 1/r_0$. If $Y = Y_0$, i.e., if the right-mover were an elementary dipole, we would have an S -matrix of order one: $S_{Y_0}(r_0) \sim 1$. The configuration which, for $Y > Y_0$, gives the result of the Kovchegov equation is the one in which the evolution of the right-mover is suppressed in such a way that, in the whole rapidity range $0 < y < Y - Y_0$, the corresponding wavefunction consists only of the parent dipole of size r_0 . This is a rare configuration, but has the advantage to involve only one dipole, and thus give a rather large contribution to S , of order one. Even though suppressed by the small probability of the particular configuration, that we shall shortly compute, this contribution remains substantially larger than that of a typical configuration, with $N(Y - Y_0) \gg 1$ gluons (cf. Eq. (24)), which is extremely small (of the same order as shown in Eq. (25)).

Let $A(r_0, Y - Y_0)$ denote the probability of the rare configuration of interest. This is the same as the survival probability of the parent dipole after a BFKL evolution over a rapidity interval $Y - Y_0$, and can be computed with the methods of Refs. [8, 10, 23, 24, 25, 26]. This probability decreases with increasing Y because of gluon radiation, and the corresponding rate is the same as the virtual term in Eq. (6):

$$\frac{\partial}{\partial y} A(r_0, y) = -\bar{\alpha}_s \int \frac{d^2 z_\perp}{2\pi} \frac{r_0^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} A(r_0, y) \quad (26)$$

with x_\perp and y_\perp labelling the quark and antiquark parts of the dipole r_0 . This equation should be integrated from $y = 0$ up to $y = Y - Y_0$. Note however that the integral is ill-defined because of poles in the integrand at $z_\perp = x_\perp$ and $z_\perp = y_\perp$. These divergences reflect the fact that one cannot avoid the emission of dipoles of arbitrarily small size.

The situation is similar to that encountered in the discussion of the Kovchegov equation in the regime where the S -matrix is small (cf. Sec. 2). Like in that case a physical cutoff, which depends on y , exists also for the problem at hand. Let us introduce this cutoff as a minimal dipole size $\rho(y)$; that is, the integral in (26) should be restricted to $(x_\perp - z_\perp)^2 > \rho^2(y)$ and $(z_\perp - y_\perp)^2 > \rho^2(y)$. Physically this means that, at rapidity y , we suppress

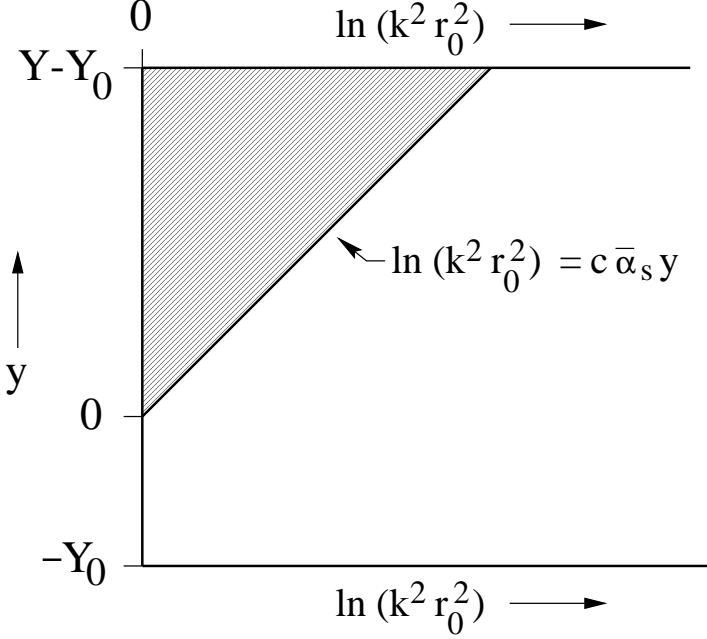


Figure 1: The configuration retained by the Kovchegov equation in the frame in which the left mover has rapidity $-Y_0$.

the radiation of dipoles with sizes larger than $\rho(y)$, but smaller dipoles are allowed³. Each of these small radiated dipoles becomes the seed of a normal BFKL evolution going from the intermediate rapidity y (at which the dipole was emitted) down to $y = 0$. The evolution is such that new dipoles are produced, and the typical size of such “children” dipoles is increasing with decreasing y . This is so because the line $\ln(r^2(y)/r_0^2) = -c\bar{\alpha}_s y$, where c is given in (11), is a line of constant amplitude for BFKL evolution [27, 28].

We see that, instead of being truly a single dipole configuration, the configuration that we are constructing also contains small dipoles, of size $\lesssim \rho(y)$ at the “moment” y of their emission, which then grow up with further decreasing y , and can become as large as $\rho^2(0) = \rho^2(y)e^{c\bar{\alpha}_s y}$ at the collision “time” $y = 0$. Still, this complicated configuration behaves, during scattering, as the idealized single dipole configuration, provided all the additional dipoles which are present in the wavefunction at $y = 0$, and hence also in the interval

³Note indeed that the dipoles which are included in the integral in eq. (26) are those whose radiation is forbidden.

$0 \leq y \leq Y - Y_0$, are much smaller than the parent dipole r_0 . (Indeed, such small dipoles undergo essentially no scattering, so the overall S -matrix for this configuration remains very close to $S_{Y_0}(r_0) \sim 1$.) The condition that $\rho(0) \lesssim r_0$ implies an upper limit on $\rho(y)$, namely $\rho^2(y) \lesssim r_0^2 e^{-c\bar{\alpha}_s y}$, which can also be written as

$$\ln(\rho^2(y)/r_0^2) \simeq -c\bar{\alpha}_s y, \quad (27)$$

since the integral on the right-hand side of (26) is dominated by the logarithmic regions of integration where $\rho(y) \ll r \ll r_0$. Here, r is the transverse size of any of the emitted dipoles, that is, either $|x_\perp - z_\perp|$, or $|y_\perp - z_\perp|$.

Thus, the dominant contribution to eq. (26) can be evaluated as:

$$\frac{\partial}{\partial y} \ln A(r_0, y) \simeq -\bar{\alpha}_s \int_{\rho^2(y)}^{r_0^2} \frac{dr^2}{r^2} = \bar{\alpha}_s \ln(\rho^2(y)/r_0^2) = -c\bar{\alpha}_s^2 y, \quad (28)$$

which after integration over y yields:

$$A(r_0, Y - Y_0) \simeq e^{-\frac{c}{2} \bar{\alpha}_s^2 (Y - Y_0)^2}. \quad (29)$$

Of course, the evolution towards larger dipoles, with sizes $r \gg r_0$, must be forbidden as well. But such large dipoles give only a small contribution to the integral in eq. (26), because of the rapid fall off of the integrand at large transverse separations $|z_\perp - b_\perp| \gg r_0$ (with $b_\perp = (x_\perp + y_\perp)/2$). Specifically, if we denote this contribution as A' (this is an additional factor which multiplies the previous contribution in eq. (29)), then A' is estimated as:

$$\frac{\partial}{\partial y} \ln A'(r_0, y) \simeq -\frac{\bar{\alpha}_s}{2} r_0^2 \int_{r_0^2} \frac{dr^2}{r^4} = -\frac{\bar{\alpha}_s}{2}.$$

After integration over y , this gives a new contribution, of order $\bar{\alpha}_s(Y - Y_0)$, to the exponent of eq. (29). This contribution is subleading in the regime of interest here, and, in any case, it goes beyond the general accuracy of the present calculation. Because of that, dipoles with sizes $r \geq r_0$ will be ignored in what follows.

The S -matrix element associated with the particular configuration of interest is finally computed as $S_Y = A(r_0, Y - Y_0)S_{Y_0}$ and coincides, as anticipated, with the result of the Kovchegov equation, Eq. (14).

In the next subsection and in Sec. 4 we shall see that a larger S -matrix can be obtained by choosing different rare configurations, which are also

frame dependent. To that end, it will be useful to have a graphical representation of the regions of evolution in the $(\ln(k_{\perp}^2 r_0^2), Y)$ plane, with k_{\perp} being the momentum conjugate to r_{\perp} . For the particular configuration that we have discussed above, the corresponding plot is displayed in Fig. 1. The shaded triangle in this plot represents the kinematical domain into which dipole emission is forbidden, and which gives the dominant contribution to A , eq. (29). Note that, up to a factor $\bar{\alpha}_s$, the exponent in eq. (29) is the same as the area of this shaded triangle.

3.5 The dominant configuration in the center of mass frame

We now turn to the task of finding those configurations that dominate very high energy dipole-dipole scattering in the CM frame. Since the typical configurations that we have previously considered, in Secs. 3.2 and 3.3, have given an S -matrix which is too small (cf. eqs. (23) and (25)), it is natural to search for configurations which are more rare in the wavefunction but which lead to a larger S -matrix. The problem with the previous calculations is that the typical configurations contain too many gluons at the time of collision, thus giving a contribution to the S -matrix which is extremely small. This suggests that the dominant configurations for evaluating S at very high energies should contain less than the mean number of gluons [8].

In view of the discussion in Sec. 3.4, the general strategy for finding the optimal configurations should be rather clear: Loosely speaking, the final goal should be to *minimize the number of gluons* by suppressing the evolution, but at the same time *maximize the probability* of the resulting configuration. The latter is largest (i.e., of order one) for the configurations subjected to normal evolution, but is rapidly decreasing for the configurations in which the evolution has been suppressed. Thus, the best results should be obtained by enforcing the *minimal suppression* which still give rise to a S -matrix of order one.

Unfortunately, since we are not able to analyze all the possible configurations, we cannot transform these considerations into a precise mathematical criterion for the search of the optimal configurations. What we can do, however, is to use the experience with the previous example in Sec. 3.4 in order to guess what should be the dominant configurations. With these configurations, we shall then evaluate the S -matrix, and find a result which is

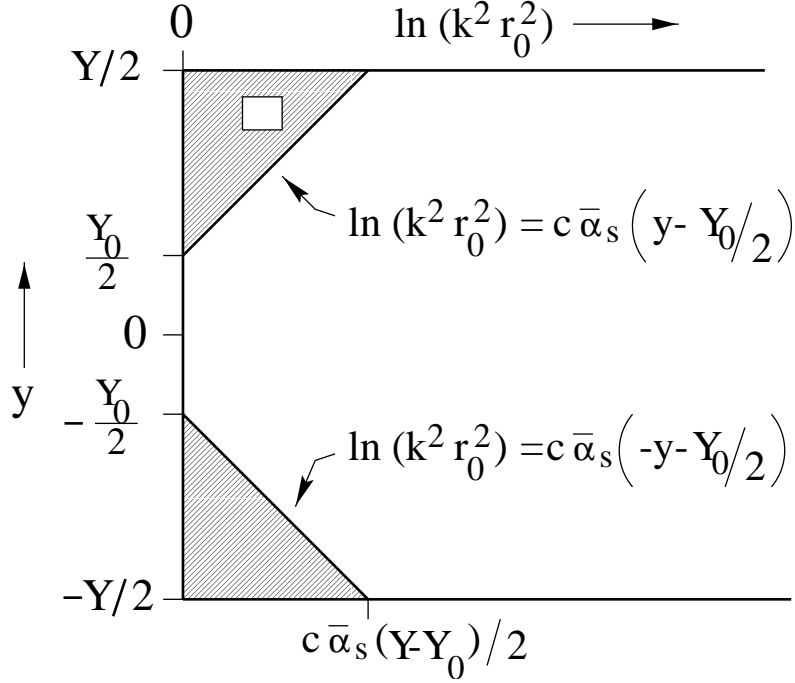


Figure 2: The optimal configuration in the center-of-mass frame.

indeed larger than in eq. (14). Finally, in later sections, we shall give arguments as to why we believe we have found the most important wavefunction configurations for evaluating S in the high energy limit.

Consider a zero impact parameter collision in the center-of-mass frame at rapidity $Y > Y_0$. (As usual, Y_0 denotes the critical value for the onset of unitarity corrections, cf. eq. (21).) For this problem, we shall require that the wavefunction of the right-moving dipole consist only of the parent dipole, of size r_0 , in the rapidity interval $Y_0/2 < y < Y/2$, with a similar requirement on the wavefunction of the left-moving dipole in the interval $-Y/2 < y < -Y_0/2$. In the rapidity intervals $-Y_0/2 < y < 0$ for the left-moving system, and, respectively, $0 < y < Y_0/2$ for the right-moving one, we allow normal, BFKL, evolution of the wavefunctions.

Of course, say in the $Y_0/2 < y < Y/2$ interval for the right-moving system, we cannot require that *all* evolution in rapidity be absent. What is required is that evolution which does exist create only very small dipoles, close to either the quark or antiquark of the parent dipole, so that in the interval $Y_0/2 < y < Y/2$ the system have no more than one dipole of size λr_0 or

larger, with λ a constant of order 1. In order to guarantee that this be the case, it is necessary to suppress the creation of dipoles much smaller than r_0 at rapidities $y > Y_0/2$; otherwise, the dipoles emitted at intermediate rapidities could evolve into dipoles of size r_0 , or larger, at rapidity $Y_0/2$. The constraints are similar to those we have just discussed in the previous subsection and are illustrated in Fig. 2. Gluon emission from the two parent dipoles, as part of the evolution which forms the left and right-moving states which scatter on each other, is forbidden if the gluon has k_\perp and y values lying in the shaded triangles⁴ of Fig. 2. (For the moment ignore the small unshaded square in the upper triangle.) The line

$$\ln(k_\perp^2 r_0^2) = c\bar{\alpha}(y - Y_0/2), \quad (30)$$

and a similar line for the lower triangle, is determined, as we have just seen, by the requirement that gluons lying to the right of that line, for any $y > Y_0/2$, cannot evolve through normal BFKL evolution to give gluons having $k_\perp \lesssim 1/r_0$ at rapidity $Y_0/2$. The constant c is again the same as in (10) and so the line (30) is a line of constant amplitude in terms of BFKL evolution. Thus evolution which crosses this line has a small probability.

Still as in Sec. 3.4, we shall denote by $A(r_0, (Y - Y_0)/2)$ the probability that the parent dipole, of size r_0 and rapidity $Y/2$, not give rise to any emissions in the upper triangle of Fig. 2. The S -matrix is then given by a factor $A(r_0, (Y - Y_0)/2)$ for each of the parent dipoles partaking in the collision, times the (partial) S -matrix for the scattering of two dipoles separated by a rapidity gap $Y_0/2 - (-Y_0/2) = Y_0$, and which are subjected both to normal evolution. The latter is, of course, $S_{Y_0}(r_0) \sim 1$. After also using eq. (29) for $A(r_0, (Y - Y_0)/2)$, one finally obtains:

$$S_Y(r_0) \simeq e^{-\frac{\varepsilon}{4} \bar{\alpha}_s^2 (Y - Y_0)^2} S_{Y_0}(r_0) \quad (31)$$

a result which is much larger than (25), and also significantly larger than the result (14) coming from the Kovchegov equation.

Thus, one can avoid a very small S -matrix, as given in (25), by not forgetting the rare parts of the wavefunction which can dominate the S -matrix when S is small. These rare configurations consist of unusually small numbers of gluons being present in the wavefunction.

⁴As discussed in Sec. 3.4, the evolution towards larger dipoles, with sizes $r \geq r_0$ (or $\ln(k_\perp^2 r_0^2) < 0$), is also forbidden, but this can be neglected in the calculation of the probability $A(r_0, (Y - Y_0)/2)$.

It should be emphasized that (31) holds only when the parent dipoles have a fixed size, r_0 in our case. For instance, if one were to scatter heavy onia states, eq. (31) would not emerge. In the case of heavy onia there is a wavefunction giving the probability of having a parent dipole of a particular size, r . Because of that, the S -matrix for the scattering of two onia at very high energies would be dominated by parent dipole sizes r of order $1/Q_s(Y)$.

3.6 Sampling other configurations

In the last section we found a particular configuration that leads to an S -matrix which is much larger than that given by the Kovchegov equation. While this is sufficient to prove that fluctuations must be important, and thus that the result in eq. (4) cannot be exact, it is, perhaps, not clear that we have found the optimal configurations for dipole-dipole scattering in the CM frame. In this section we give arguments to the end that (31) is the correct answer and that the configurations we have focused on are indeed the dominant ones.

The first change one may consider in the previous calculation would be to modify the values of the intermediate rapidities $\pm Y_0/2$ which separate between suppressed and normal evolution. So, let us replace $Y_0 \rightarrow Y_1$, with $Y_1 \neq Y_0$, in Fig. 2 and the related calculations. There are two possibilities — $Y_1 < Y_0$, or $Y_1 > Y_0$ —, but as we argue now they both lead to a S -matrix which is much smaller than that in eq. (31). *i)* If $Y_1 < Y_0$, the probability $A(r_0, (Y - Y_1)/2)$ for the suppressed configuration is even smaller than before, while the partial S -matrix S_{Y_1} for the collision between two normally evolved systems is still of order one (since $Y_1 < Y_0$ corresponds to weak scattering). Thus, this situation is clearly less favourable. *ii)* If $Y_1 > Y_0$, the probability $A(r_0, (Y - Y_1)/2)$ is larger, but this gain is more than compensated by the strong decrease in the partial S -matrix S_{Y_1} . To see this, note that for the calculation of S_{Y_1} we are either in the situation described in Sec. 3.2 (if $Y_1 < 2Y_0$), or in that of Sec. 3.3 (if $Y_1 > 2Y_0$), and, as we have seen, both these situations give only tiny contributions to S . If we focus on $|\ln S_Y|$, for definiteness, then $|\ln S_{Y_1}|$ grows exponentially with the difference $Y_1 - Y_0$ (cf. eqs. (22) or (25)), while the corresponding decrease in $|\ln A|$ is only quadratic. Clearly, the overall S -matrix decreases rapidly with increasing $Y_1 - Y_0$.

The other major change that we could make in our calculation in the last section would be to allow some limited emission of gluons into the forbidden

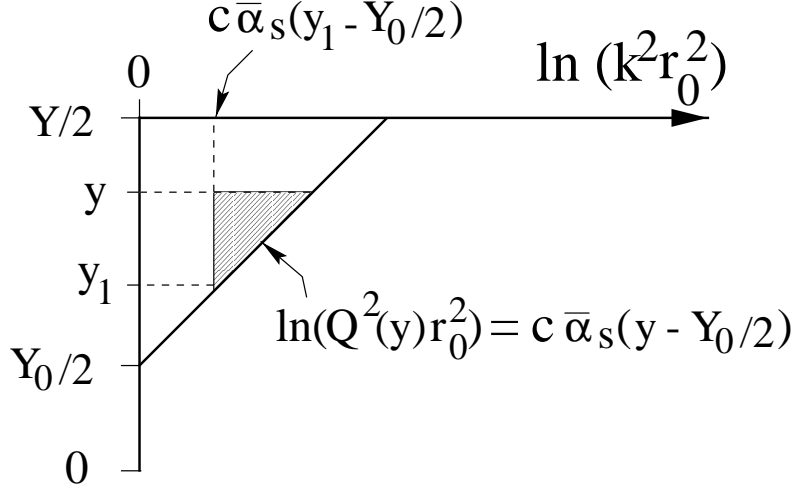


Figure 3: The shaded area is the domain of suppressed evolution for a parent dipole within the small unshaded rectangle in Fig. 2.

triangles of Fig. 2. Recall that the area of such a triangle determines the penalty factor A associated with suppressing the evolution (cf. the remark at the end of Sec. 3.4). Then let us calculate the change in our previous result which occurs if we allow emission into the small square in the upper triangle of Fig. 2. Take the area of the square to be $1/\bar{\alpha}_s$, so that now one gets a factor A which is a factor of e larger than the previous one:

$$A' = e \times A(r_0, (Y - Y_0)/2), \quad (32)$$

with $A(r_0, \frac{Y-Y_0}{2})$ given by eq. (29). We have less suppression than before because we have *not* forbidden the parent dipole to emit into the small square. However, now we must forbid the newly created (small) dipole from evolving further and again creating a strong suppression like that given in (25). In Fig. 3, the shaded triangle represents the region into which one must forbid the additional dipole from emitting gluons⁵. If y and k_\perp are the coordinates of the gluon emitted by the parent dipole (i.e., the central coordinates of the small unshaded square in Fig. 2), and if y_1 is defined by (see Fig. 3):

$$\ln(k_\perp^2 r_0^2) = c\bar{\alpha}_s(y_1 - Y_0/2), \quad (33)$$

⁵Once again, we consider only that forbidden area which gives the dominant contribution to the probability A ; see eq. (35).

then the shaded area in Fig. 3 — the area into which emission from the secondary dipole must now be forbidden — is

$$a = \frac{1}{2} c \bar{\alpha}_s (y - y_1)^2. \quad (34)$$

The probability of not emitting into this area is:

$$A'' = e^{-\bar{\alpha}_s a} = e^{-\frac{c}{2} \bar{\alpha}_s^2 (y - y_1)^2}. \quad (35)$$

In order for this suppression not to be larger than the gain from being allowed to have the secondary emission, cf. eq. (32), one must require that $\bar{\alpha}_s a < 1$. This implies the following condition on the location of the small square in Fig. 2 :

$$\left[\ln(Q^2(y)r_0^2) - \ln(k_\perp^2 r_0^2) \right]^2 < 2c, \quad (36)$$

where, for more clarity, we have denoted the rightmost borderline of the suppressed area in Fig. 3 as (cf. eq. (30)): $\ln(Q^2(y)r_0^2) = c\bar{\alpha}(y - Y_0/2)$.

Eq. (36) shows that, for a given y , the point $\ln(k_\perp^2 r_0^2)$ lies within one unit of the boundary, $\ln(Q^2(y)r_0^2)$. But since the boundary is ambiguous at this level of precision — e.g., the right hand side of eq. (30) is specified only up to corrective terms of order one —, we see that it does not seem possible to relax the assumption we made in deriving eq. (31), namely that there must be no emission into the triangle regions of Fig. 2.

4 The picture in a general frame

In this section we show how the result (31) comes about in an arbitrary frame. In order to describe the frame choice and the scattering picture, it is useful to refer to Fig. 4. We scatter a left-moving parent dipole of size r_1 and rapidity $-Y_2$ on a right-moving parent dipole having size r_0 and rapidity $Y - Y_2$. Recall that Y_0 is the value of the rapidity difference between two such dipoles where the S -matrix begins to be significantly different from one. We also suppose that $Y_2 \leq \frac{1}{2}(Y - Y_0)$, for later convenience.

Clearly one cannot allow both dipoles r_0 and r_1 to have any significant amount of normal BFKL evolution, or else we will find a strong suppression of the type given by (25). Since the left-moving dipole has the smaller rapidity it is natural (and easier) to suppress its evolution, and we will describe

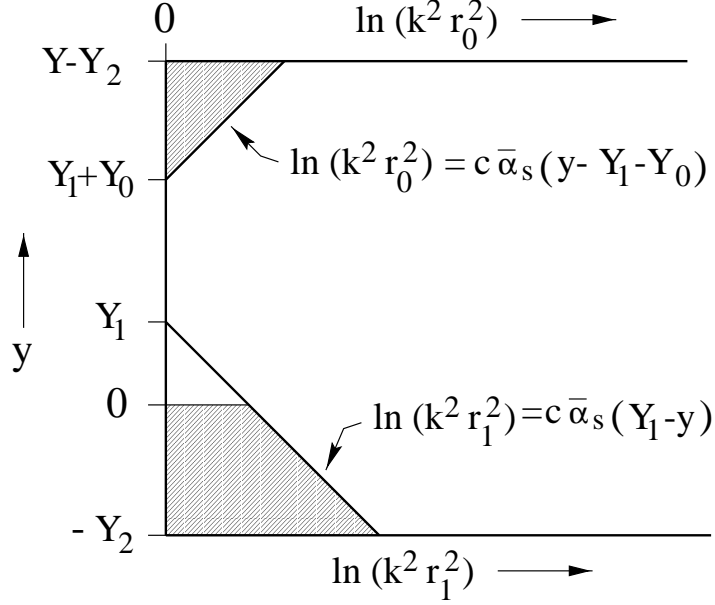


Figure 4: Choosing the optimal configuration in some generic frame.

that suppression in a moment. As for the dipole r_0 , we suppress evolution over its $Y - Y_2 - (Y_1 + Y_0)$ highest units of rapidity exactly as we did in Sec. 3, with the region of suppression given by the upper shaded triangle of Fig. 4. In the lowest $Y_0 + Y_1$ units of rapidity the parent dipole undergoes normal evolution. We will determine Y_1 later, by maximizing the S -matrix. The unshaded triangle, whose rapidity values go from 0 to Y_1 , denotes the saturation region for the right-mover, that is, the region where the parent dipole r_0 has evolved into a Color Glass Condensate (so the evolution is non-linear within that particular region). Let $Q_s \equiv Q_s(Y_0 + Y_1)$ be the saturation scale which characterizes this condensate. Since obtained after a normal evolution through $Y_0 + Y_1$ units of rapidity, this is of the form:

$$Q_s^2(Y_0 + Y_1) = Q_0^2 e^{c\bar{\alpha}(Y_0+Y_1)}, \quad (37)$$

where Q_0^2 is an intrinsic scale of the right-moving system, proportional to $1/r_0^2$ and, possibly, powers of α_s . By the very definition of Y_0 , we have $Q_s^2(Y_0) = 1/r_1^2$, so the saturation momentum (37) can be rewritten as:

$$Q_s^2 \equiv Q_s^2(Y_0 + Y_1) = \frac{1}{r_1^2} e^{c\bar{\alpha}Y_1}, \quad (38)$$

where the dependence upon the unknown scale Q_0^2 has disappeared.

Turning now to the parent dipole r_1 , we require that no additional dipoles be created, through gluon emission, which would have a strong interaction with the right-moving Color Glass Condensate. Clearly, the dipoles which would have such strong interactions are those which, at the time of scattering (i.e., at $y = 0$), would be of size $r \gtrsim 1/Q_s$. This means that, at all the intermediate rapidities within the range $-Y_2 < y < 0$, we have to suppress the emission of those dipoles which, after a normal evolution over the last $|y|$ units of rapidities, could become of size $1/Q_s$, or larger. By referring to the similar discussion in Sec. 3.4, we see that the maximal size $\rho(y)$ allowed for a dipole emitted at rapidity y is constrained by

$$\rho^2(y) \lesssim \frac{1}{Q_s^2} e^{-c\bar{\alpha}_s|y|} = r_1^2 e^{-c\bar{\alpha}_s(Y_1-y)}, \quad (39)$$

where we have also used eq. (38) together with the fact that $|y| = -y$. In terms of the conjugate momentum variable k_\perp , this gives the lower shaded region in Fig. 4 as the region into which radiation is forbidden.

We are now in a position to estimate the S -matrix as

$$S_Y(r_0, r_1) = A_R(r_0, Y - Y_0 - Y_1 - Y_2) \mathcal{S}_{Y_0+Y_1}(r_0, r_1) A_L(r_1, Y_2), \quad (40)$$

where A_R and A_L are the suppression factors from the no emission requirements for the two dipoles, and $\mathcal{S}_{Y_0+Y_1}(r_0, r_1)$ is the S -matrix for scattering an elementary dipole of size r_1 on a dipole r_0 which has evolved into a Color Glass Condensate characterized by the saturation momentum (38).

To compute \mathcal{S} , one can rely on the Kovchegov equation, or, more exactly, on its approximate form in eq. (9), valid deeply at saturation. This is correct for the present purpose since, as discussed at the beginning of Sec. 3.4, the Kovchegov equation describes correctly the scattering between an elementary dipole (here, the dipole r_1) and the typical configurations in a Color Glass Condensate (here, the dipole r_0 which has been allowed to carry out its normal evolution over the rapidity interval $Y_0 + Y_1$). By using eq. (14), one obtains:

$$\mathcal{S}_{Y_0+Y_1}(r_0, r_1) \simeq e^{-\frac{c}{2} \bar{\alpha}_s^2 Y_1^2} S_{Y_0}(r_0). \quad (41)$$

We note in passing that the exponent in (38) is, except for a factor of $\bar{\alpha}_s$, the area of the unshaded triangle in Fig. 4, and where, as usual, the exponents

of A_R and A_L are given in terms of the area of the upper and lower shaded regions of Fig. 4. Thus

$$A_R = e^{-\frac{c}{2} \bar{\alpha}_s^2 (Y - Y_2 - Y_1 - Y_0)^2}, \quad (42)$$

and, similarly,

$$A_L = e^{-\frac{c}{2} \bar{\alpha}_s^2 [(Y_1 + Y_2)^2 - Y_1^2]}. \quad (43)$$

By combining eqs. (40)–(43), one obtains:

$$S_Y = \exp \left\{ -\frac{c}{2} \bar{\alpha}_s^2 [(Y - Y_2 - Y_1 - Y_0)^2 + (Y_1 + Y_2)^2] \right\} S_{Y_0}(r_0). \quad (44)$$

The S -matrix given by (44) refers to a particular set of configurations of the wavefunctions, characterized by one parameter: the rapidity Y_1 which describes the amount of evolution in the right-moving system. (Recall that the other rapidity parameter in eq. (44), namely Y_2 , specifies the frame of reference.) The actual S -matrix should be determined by the value of Y_1 which maximizes the right hand side of eq. (44), or, equivalently, which minimizes the exponent there. This condition yields:

$$Y_1 = \frac{1}{2} (Y - Y_0) - Y_2. \quad (45)$$

This is non-negative, because of our initial assumption $Y_2 \leq \frac{1}{2}(Y - Y_0)$.

Finally, S_Y is obtained by evaluating eq. (44) with the optimal value for Y_1 , eq. (45) :

$$S_Y(r_0, r_1) = e^{-\frac{c}{4} \bar{\alpha}_s^2 (Y - Y_0)^2} S_{Y_0}(r_0). \quad (46)$$

(The right hand side depends upon r_1 via the “critical” rapidity Y_0 .) This result is independent of Y_2 (i.e., upon the choice of a frame), and is exactly the same as the corresponding result in the CM frame, eq. (31).

In fact, the picture of the evolution for the optimal configuration in the CM frame, as shown previously in Fig. 2, can be obtained from the general picture in Fig. 4 by first choosing Y_2 in such a way that $Y_1 = 0$ — this requires $Y_2 = (Y - Y_0)/2$ (cf. eq. (45)) —, and then performing a supplementary boost by a rapidity amount $\Delta Y = -Y_0/2$ (so that $Y_2 \rightarrow Y/2$, as in Fig. 2). The latter operation changes only the position of 0 on the rapidity axis, but not also the picture of the evolution⁶.

⁶Indeed, if $Y_1 = 0$, there is no non-linear evolution involved, but only linear, BFKL, evolution, which is invariant under a small boost.

In particular, it is instructive to visualize the evolution in the “dipole frame”, i.e., the rest frame of the left-mover ($Y_2 = 0$), in which both the optimal configuration (cf. Fig. 4) and the configuration retained by the Kovchegov equation are explicitly known. These configurations are illustrated in Figs. 5.a and b, respectively. As usual, a shaded area represents a kinematical domain into which dipole emission is forbidden, while an unshaded triangle represents the saturation region. Also, up to a factor of $\bar{\alpha}_s$, the total area of the (shaded and unshaded) triangles represents the exponent of the S -matrix associated with the respective configuration. As manifest on these figures, the area of the empty triangle in Fig. 5.b is twice as much as the sum of the areas of the two small triangles in Fig. 5.a.

Note also that, for both configurations in Fig. 5, the right-mover ends up as a Color Glass Condensate at $y = 0$. But while the configuration retained by the Kovchegov equation (cf. Fig. 5.b) corresponds to normal evolution over the whole rapidity range — BFKL evolution from $y = Y$ down to $y = Y - Y_0$, and then non-linear, JIMWLK, evolution over the remaining $Y - Y_0$ units of rapidity —, in the optimal configuration (cf Fig. 5.a), the normal evolution is allowed only over the lower $(Y + Y_0)/2$ units of rapidity. As a result, the saturation scale which characterizes the condensate in the optimal configuration, namely, $Q_s((Y + Y_0)/2)$, is much smaller than the corresponding scale $Q_s(Y)$ for the normal evolution.

Acknowledgments

Much of this work was done while one of the authors (A.M.) was a visitor at LPT (Université de Paris XI, Orsay). He wishes to thank Dr. Dominique Schiff for her hospitality and support during this visit.

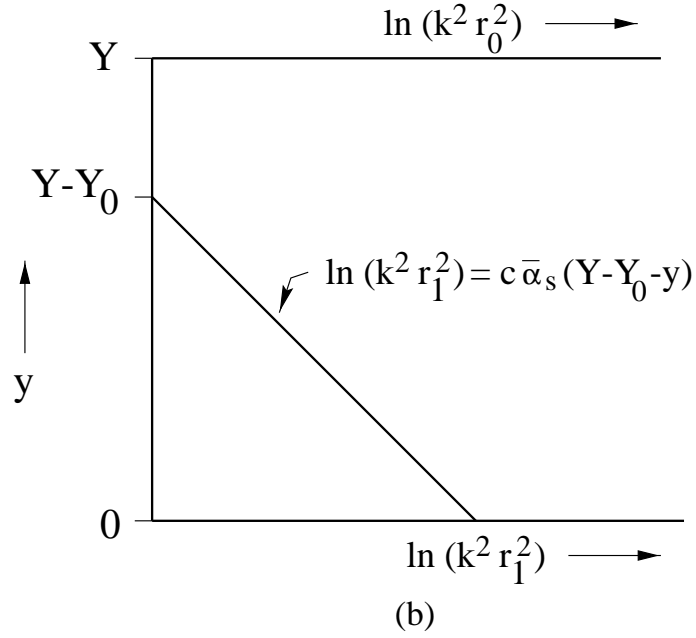
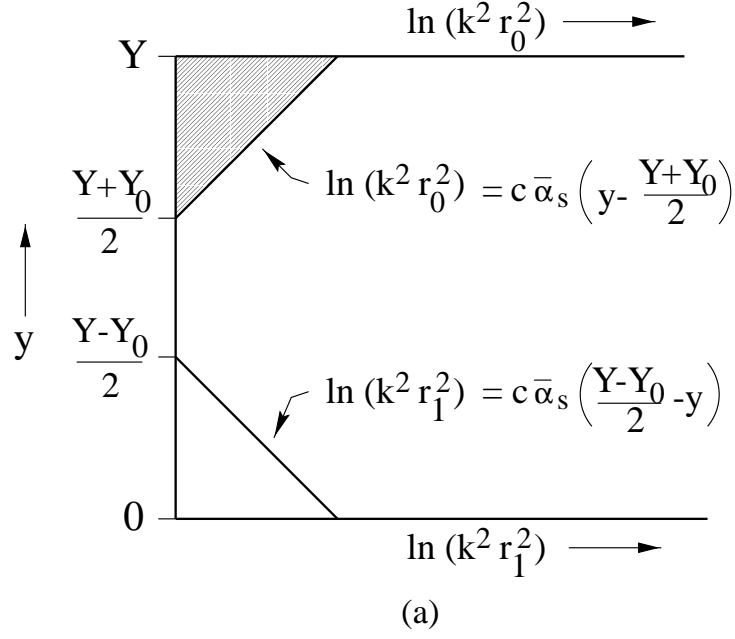


Figure 5: The optimal configuration (above) and the configuration retained by the Kovchegov equation (below) in the frame in which $Y_2 = 0$.

References

- [1] E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. **A692** (2001) 583; Phys. Lett. **B510** (2001) 133.
- [2] E. Ferreira, E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. **A703** (2002) 489.
- [3] E. Iancu and L. McLerran, Phys. Lett. **B510** (2001) 145.
- [4] For a review see E. Iancu and R. Venugopalan, hep-ph/0303204.
- [5] L.V. Gribov, E.M. Levin and M.G. Ryskin, Phys. Rep. **100** (1983) 1.
- [6] A.H. Mueller, and J. Qiu, Nucl. Phys. **B268** (1986) 427.
- [7] Yu. V. Kovchegov, Phys. Rev. **D60** (1999) 034008; Phys. Rev. **D61** (2000) 074018.
- [8] A.H. Mueller and G. P. Salam, Nucl. Phys. **B475** (1996) 293.
- [9] E. Levin and K. Tuchin, Nucl. Phys. **B573** (2001) 833; Nucl. Phys. **A691** (2001) 779.
- [10] G.P. Salam, Nucl. Phys. **B461** (1996) 512.
- [11] I.I. Balitsky, Nucl.Phys. **B463** (1996) 99; Phys. Rev. Lett. **81** (1998) 2024; Phys. Lett. **B518** (2001) 235.
- [12] J.Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Nucl. Phys. **B504** (1997) 415; Phys. Rev. **D59** (1999) 014014.
- [13] H. Weigert, Nucl. Phys. **A703** (2002) 823.
- [14] N. Armesto and M.A. Braun, Eur. Phys. J. **C20** (2001) 517.
- [15] E. Gotsman, E.M. Levin, M. Lublinsky and U. Maor, Nucl. Phys. **A696** (2001) 851.
- [16] K. Golec-Biernat, L. Motyka and A.M. Stasto, Phys. Rev. **D65** (2002) 07437.

- [17] E. Levin and M. Lublinsky, *Phys. Lett.* **B521** (2001) 233; M. Lublinsky, *Eur. Phys. J.* **C21** (2001) 513.
- [18] K. Golec-Biernat and A.M. Stasto, *Nucl. Phys.* **B668** (2003) 345.
- [19] J. Albacete, N. Armesto, A. Kovner, C. Salgado and U. Wiedemann, hep-ph/0307179.
- [20] A. H. Mueller, *Nucl. Phys.* **B558** (1999) 285.
- [21] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *Sov. Phys. JETP* **45** (1977) 199.
- [22] Ya. Ya. Balitsky and L.N. Lipatov, *Sov. J. Nucl.Phys.* **28** (1978) 822.
- [23] A.H. Mueller, *Nucl. Phys.* **B415** (1994) 373.
- [24] A.H. Mueller and B. Patel, *Nucl. Phys.* **B425** (1994) 471.
- [25] A.H. Mueller, *Nucl. Phys.* **B437** (1995) 107.
- [26] E. Iancu and A.H. Mueller, hep-ph/0308315.
- [27] E. Iancu, K. Itakura, and L. McLerran, *Nucl. Phys.* **A708** (2002) 327.
- [28] A. H. Mueller and D.N. Triantafyllopoulos, *Nucl. Phys.* **B640** (2002) 331; D.N. Triantafyllopoulos, *Nucl. Phys.* **B648** (2002) 293.